# Math 166 Test 2

Fall 2018, Prof. Hee Jung Kim

**Note: No books, No any type of calculators. You must show all your work in the blue books. You must show enough work to indicate how you get your answer. You will lose credit for incorrect statements or incorrect mathematical expressions. Neatness and clarity are important. Do not give two or more answers for the same problem. Cross out any scratch work, or label it as scratch. Return used blue books for any scratch work and a note sheet if you have.**

1. [20 pt] Find the following indefinite integral.

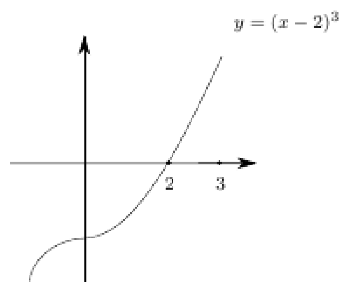
   (a) $\displaystyle\int \frac{x^2}{(1-4x^2)^{3/2}}dx$ (b) $\displaystyle\int \frac{2x^2-x+9}{x^3+9x}dx$

2. [20 pt] Determine whether or not the following integrals converge.

   (a) $\displaystyle\int_0^{\pi/6} \frac{1}{x^4+x^2}dx$ (b) $\displaystyle\int_{-\infty}^{\infty} x\,dx$

3. [20 pt] Find the exact arc length $s$ of the curve $y = x^2 - \frac{1}{8}\ln x$ over $[1, e^8]$.

4. [20 pt] Find the area of the surface generated by rotating the curve given by $y = (x-2)^3$ over $[2,3]$ about the $x$-axis.




5. [20 pt] Evaluate the following integral.

$$\int_0^{\infty} \frac{8e^{-x}}{e^{-2x}-16}dx$$

1. Solve the following integrals:

$$a.) \int \frac{x^2\,dx}{(1-4x^2)^{3/2}} \qquad\qquad b.) \int \frac{2x^2-x+9}{x^3+9x}\,dx$$

1.a.) Regarding the first one all this is a trig sub into a tricky trig identity.

$$\begin{aligned}
\int \frac{x^2\,dx}{(1-4x^2)^{3/2}} &= \int \frac{(0.5\sin\theta)^2(0.5\cos\theta)\,d\theta}{(1-4(0.5\sin\theta)^2)^{3/2}} && x = 0.5\sin\theta \\
&= \frac{1}{8}\int \frac{\sin^2\theta\cos\theta}{(1-\sin^2\theta)^{3/2}}\,d\theta && dx = 0.5\sin\theta\,d\theta \\
&= \frac{1}{8}\int \frac{\sin^2\theta\cos\theta}{(\cos^2\theta)^{3/2}}\,d\theta && 1-\sin^2\theta = \cos^2\theta \\
&= \frac{1}{8}\int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}\,d\theta \\
&= \frac{1}{8}\int \tan^2\theta\,d\theta && \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} \\
&= \frac{1}{8}\int \sec^2\theta - 1\,d\theta \\
&= \frac{\tan\theta}{8} - \frac{\theta}{8} + c \\
&= \frac{2x}{8\sqrt{1-4x^2}} - \frac{\arcsin(2x)}{8} + c \\
&= \frac{x}{4\sqrt{1-4x^2}} - \frac{\arcsin(2x)}{8} + c
\end{aligned}$$

1.b.) Regarding the second, this is merely a partial fractions problem with an simple

$$\begin{aligned}
\int \frac{2x^2-x+9}{x^3+9x}\,dx &= \int \frac{2x^2-x+9}{x(x^2+9)}\,dx \\
&= \int \frac{A}{x} + \frac{Bx+C}{x^2+9}\,dx && \text{for some } A,B,C \in \mathbb{R} \\
&= \int \frac{A}{x} + \frac{Bx}{x^2+9} + \frac{C}{x^2+9}\,dx \\
&= A\ln x + \frac{B\ln|x^2+9|}{2} + \frac{C}{3}\arctan\left(\frac{x}{3}\right) + c && \text{Integration constant } c
\end{aligned}$$

Discerning what $A, B, C \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{2x^2-x+9}{x^3+9x} &= \frac{A}{x} + \frac{Bx+C}{x^2+9} \\
2x^2-x+9 &= A(x^2+9) + (Bx+C)x \\
2x^2-x+9 &= (A+B)x^2 + Cx + 9A \\
&\implies 9A = 9 \implies A = 1 \\
&\implies C = -1 \\
&\implies A+B = 2 \implies B = 1
\end{aligned}$$

Therefore, we have the solution

$$\int \frac{2x^2-x+9}{x^3+9x}\,dx = \ln x + \frac{\ln|x^2+9|}{2} - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$$

2. Determine whether or not the following improper integrals converge:

$$a.)\ \int_0^{\pi/6} \frac{dx}{x^4+x^2} \qquad\qquad b.)\ \int_{-\infty}^{\infty} x\,dx$$

2.a) Comparing this to the $p$-integral

$$\int_R^{\pi/6} \frac{dx}{x^4+x^2} \geq \int_R^{\pi/6} \frac{dx}{2x^2}$$

We know that the R.H.S. diverges as $R \to 0^+$ since $p > 1$, therefore the integral on the L.H.S. also diverges.

2.b) For the second one we know that since $\lim_{x\to\infty} x \neq 0$ the divergence test says this diverges. Alternatively, try looking at Webwork 5 #7 for inspiration.

3. Compute the arclength of $f(x) := x^2 - \frac{\ln x}{8}$ on the interval $[1, e^8]$.
First computing the derivative we get

$$f'(x) = 2x - \frac{1}{8x}$$

This gives the following, after plugging into the arclength formula

$$\begin{aligned}
\int_a^b \sqrt{1+(f'(x))^2}\,dx &= \int_1^{e^8} \sqrt{1+\left(2x-\frac{1}{8x}\right)^2}\,dx \\
&= \int_1^{e^8} \sqrt{1+\left(2^2x^2 - 2\frac{2x}{8x} + \frac{1}{8^2x^2}\right)}\,dx \\
&= \int_1^{e^8} \sqrt{1+2^2x^2 - \frac{1}{2} + \frac{1}{8^2x^2}}\,dx \\
&= \int_1^{e^8} \sqrt{2^2x^2 + \frac{1}{2} + \frac{1}{8^2x^2}}\,dx \\
&= \int_1^{e^8} \sqrt{\left(2x+\frac{1}{8x}\right)^2}\,dx \\
&= \int_1^{e^8} 2x + \frac{1}{8x}\,dx \\
&= \left[x^2 + \frac{\ln x}{8}\right|_{x=1}^{e^8} \\
&= \left((e^8)^2 + \frac{\ln(e^8)}{8}\right) - \left(1^2 + \frac{\ln 1}{8}\right) \\
&= e^{16} + 1 - 1 = e^{16}
\end{aligned}$$

4. Compute the surface area of the function $f(x) := (x-2)^3$ revolved about the $x$-axis on the interval $[2, 3]$.
Finding the derivative gives by chain rule we get$f'(x) = 3(x-2)^2$. Plugging

this into the formula for surface area we get:

$$\begin{aligned}
\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx &= \pi \int_2^3 (x-2)^3\sqrt{1+(3(x-2)^2)^2}\,dx \\
&= \pi \int_2^3 (x-2)^3\sqrt{1+9(x-2)^4}\,dx && u = 1+9(x-2)^4 \\
&= \pi \int_1^{10} \frac{\sqrt{u}}{36}\,dx && du = 36(x-2)^3\,dx \\
&= \pi \left[\frac{\sqrt{u^3}}{2\cdot 3^3}\right|_{u=1}^{10} \\
&= \frac{\pi\sqrt{10^3}}{2\cdot 3^3} - \frac{\pi\sqrt{1^3}}{2\cdot 3^3} \\
&= \frac{10\pi\sqrt{10}-\pi}{2\cdot 3^3}
\end{aligned}$$

5. Compute the following integral

$$\int_0^\infty \frac{8e^{-x}}{e^{-2x}-16}\,dx$$

This question is most easily seen by first doing a u-sub then following that up by integration by parts. Since the only problem area we have is the upper bound, we may compute the integral as an aside first then worry about the bound later,

$$\begin{aligned}
\int \frac{8e^{-x}}{e^{-2x}-16}\,dx &= \int \frac{8e^{-x}}{e^{-2x}-16}\,dx && u = e^{-x} \\
&= -\int \frac{8}{u^2-16}\,du && du = -e^{-x}dx \\
&= -\int \frac{8}{(u-4)(u+4)}\,du \\
&= -\int \frac{1}{u-4} - \frac{1}{u+4}\,du && \text{partial fractions (do it)} \\
&= -(\ln|u-4| - \ln|u+4|) + c \\
&= \ln|u+4| - \ln|u-4| + c \\
&= \ln|e^{-x}+4| - \ln|e^{-x}-4| + c \\
&= \ln\left|\frac{e^{-x}+4}{e^{-x}-4}\right| + c
\end{aligned}$$

Finally computing the definite integral we have

$$\begin{aligned}
\int_0^\infty \frac{8e^{-x}}{e^{-2x}-16}\,dx &= \lim_{R\to\infty}\int_0^R \frac{8e^{-x}}{e^{-2x}-16}\,dx \\
&= \lim_{R\to\infty}\left[\ln\left|\frac{e^{-x}+4}{e^{-x}-4}\right|\right|_{x=0}^{R} \\
&= \lim_{R\to\infty}\ln\left|\frac{e^{-x}+4}{e^{-x}-4}\right| - \ln\left|\frac{e^0+4}{e^0-4}\right| \\
&= \ln\left|\lim_{R\to\infty}\frac{e^{-x}+4}{e^{-x}-4}\right| - \ln\left|\frac{5}{3}\right| \qquad \text{If } \frac{e^{-x}+4}{e^{-x}-4}\to c\neq 0 \\
&= \ln\left|\frac{0+4}{0-4}\right| - \ln\left|\frac{5}{3}\right| \\
&= \ln\left|\frac{3}{5}\right|
\end{aligned}$$